

Achieving Cluster Consensus in Continuous-Time Networks of Multi-Agents With Inter-Cluster Non-Identical Inputs

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Abstract

In this paper, cluster consensus in continuous-time networks of multi-agents with time-varying topologies via non-identical inter-cluster inputs is studied. The cluster consensus contains two aspects: intra-cluster synchronization, that the state differences between agents in the same cluster converge to zero, and inter-cluster separation, that the states of the agents in different clusters do not approach. δ -cluster-spanning-tree in continuous-time networks of multi-agent systems plays essential role in analysis of cluster synchronization. Inter-cluster separation can be realized by imposing adaptive inputs that are identical within the same cluster but different in different clusters, under the inter-cluster common influence condition. Simulation examples demonstrate the effectiveness of the derived theoretical results.

Index Terms

Cluster consensus, multi-agent system, cooperative control, linear system

I. INTRODUCTION

Consensus problems of multi-agent systems have attracted broad attentions from various contexts (see [1]-[3]). In general, the main objective of consensus problems is to make all

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agents converge to some common state by designing proper algorithms. For this purpose, various consensus algorithms have been proposed [4]-[8].

The results of almost all previous works were concerned with consensus with a common consistent state, while we are considering cluster consensus, i.e., agents in networks are divided into several disjoint groups, called *clusters*, in the sense that all agents in the same cluster completely synchronize but the dynamics in different clusters does not coincide. In reality, a number of practical models can be transformed into this cluster consensus problem, for instance, social learning network under different environments [9]. Social learning focuses on the opinion dynamics in the society, in which individuals engage in communication with their neighbors in order to learn from their experiences. Consider that the belief of each individual is affected by different religious beliefs or cultural backgrounds. This affection flags the clusters that each individual belongs to.

In [10]-[15], the authors considered cluster (group) synchronization (consensus) problems of networks with multi-agents. In [10], [11], for linearly coupled multi-agents systems, the authors derived conditions on coupling matrix to guarantee group consensus (intra-cluster synchronization), but the inter-cluster separation was not considered. In [12], agents in different clusters have different dynamics of uncoupled node systems, the inter-cluster separation was not proved rigorously (but only assumed). Since it is quite difficult to prove inter-cluster separation for general nonlinear coupled systems (up to now, no way to prove). In [14], the dynamics of nodes are special, hence, the final states of agents can be given directly. In this paper, the inter-cluster separation is actually one of main aims, which is realized by imposing the inter-cluster different, intra-cluster identical inputs.

In our previous paper [9], we investigated cluster consensus problem in discrete-time networks of multi-agents, which provided the basic ideas. However, There still is big difference between discrete-time networks and continuous-time system. In addition, in comparison with [9], in the present paper, the static inter-cluster influence matrix in [9] is replaced by time-varying inter-cluster influence matrix sequence; the assumption of existence of self-links in [9] are removed; the formation of inputs to a more general scenario are extended, while [9] considered that the inputs among different clusters only differ by a proportionality constant. Finally, the concepts relating graph theory are generalized, too. For example, we propose " δ -cluster-spanning-tree across time interval \mathcal{T} " (see below).

II. PRELIMINARIES

In this section, we present some necessary notations and definitions of graph and matrix theory. For more details, we refer readers to textbooks [17], [18].

For a matrix L , denote L_{ij} the element of L on the i -th row and j -th column. L^\top denotes the transpose of L . E_n and O_n denote the n -dimensional identity matrix and zero matrix. $\mathbf{1}$ denotes the column vector whose components all equal to 1 and $\mathbf{0}$ denotes the column vector whose components all equal to 0. $\|z\|$ denotes a vector norm of a vector z and $\|L\|$ denotes the matrix norm of L induced by the vector norm $\|\cdot\|$.

An $n \times n$ matrix A is called a *stochastic matrix* if $A_{ij} \geq 0$ for all i, j , and $\sum_{j=1}^n A_{ij} = 1$ for $i = 1, \dots, n$. An $n \times n$ matrix L is called a *Metzler matrix with zero row sums* if $L_{ij} \geq 0$ and $\sum_{j=1}^n L_{ij} = 0$ holds for all $i \neq j$, $i = 1, \dots, n$.

A directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$ consists of a vertex set $\mathcal{V} = \{v_1, \dots, v_n\}$, a directed edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, i.e., an edge is an ordered pair of vertices in \mathcal{V} . A (directed) *path* of length l from vertex v_j to v_i , denoted by $(v_{r_1}, \dots, v_{r_{l+1}})$, is a sequence of $l+1$ distinct vertices with $v_{r_1} = v_i$ and $v_{r_{l+1}} = v_j$ such that $(v_{r_k}, v_{r_{k+1}}) \in \mathcal{E}$. We say that \mathcal{G} has self-links if $(v_i, v_i) \in \mathcal{E}$ for all $v_i \in \mathcal{V}$.

An $n \times n$ nonnegative matrix A can be associated with a directed graph $\mathcal{G}(A)$ in such a way that $(v_i, v_j) \in \mathcal{E}(\mathcal{G}(A))$ if and only if $A_{ij} > 0$. Similarly, for a Metzler matrix L , it is associated with a graph without self-links, denoted by $\mathcal{G}(L)$.

Definition 1: [9] For a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a *clustering* \mathcal{C} is defined as a disjoint division of the vertex set, namely, a sequence of subsets of \mathcal{V} , $\mathcal{C} = \{\mathcal{C}_1, \dots, \mathcal{C}_K\}$, that satisfies: (1) $\bigcup_{p=1}^K \mathcal{C}_p = \mathcal{V}$; (2) $\mathcal{C}_k \cap \mathcal{C}_l = \emptyset$, $k \neq l$.

Consider the following continuous-time system with external adapted inputs:

$$\dot{x}_i(t) = \sum_{j=1}^n L_{ij}(t)[x_j(t) - x_i(t)] + I_i(t), \quad i = 1, \dots, n \quad (1)$$

where $t \in \mathbb{R}^+ = [0, \infty)$ and $x_i(t) \in \mathbb{R}$ denotes the state variable of the agent i , $L_{ij}(t) \geq 0$ denotes the coupling weight from agent j to i , $I_i(t)$, $i = 1, \dots, n$ are external scalar inputs. Let $L_{ii}(t) = -\sum_{j=1, j \neq i}^n L_{ij}(t)$, then for each $t > 0$, the connection matrix $L(t) = [L_{ij}(t)]_{i,j=1}^n$ is a Metzler matrix with zero row sum. The matrix $L(t)$ is associated with a time-varying graph $\mathcal{G}(L(t))$.

For systems with switching topologies, some researchers introduce the concept of dwell time, which is a pre-specified positive constant to describe the time length staying in current topology, i.e., in some time interval $[t_1, t_2]$, $L(t) = L$ are constant. In this paper, we don't make this assumption. By using the concept of δ -edge [16], we transform the continuous-time case to the discrete case with some sophisticated analysis.

Definition 2: $\mathcal{G}(L(t))$ is said to have a δ -edge from vertex v_j to v_i across $[t_1, t_2]$, if $\int_{t_1}^{t_2} L_{ij}(t)dt > \delta$. For a given clustering $\mathcal{C} = \{\mathcal{C}_1, \dots, \mathcal{C}_K\}$, $L(t)$ has a δ -cluster-spanning-tree across $[t_1, t_2]$ (w.r.t. \mathcal{C}) if each cluster \mathcal{C}_p , $p = 1, \dots, K$, has a vertex $v_p \in \mathcal{V}$ and a δ -path (path composed of δ -edges) from v_p to all vertices in \mathcal{C}_p across $[t_1, t_2]$.

It should be pointed out that the root of \mathcal{C}_p and the paths from the root to the vertices in \mathcal{C}_p do not necessarily in \mathcal{C}_p ; the root vertex of a cluster is unnecessarily identical with roots in other clusters.

Definition 3: For a given clustering $\mathcal{C} = \{\mathcal{C}_1, \dots, \mathcal{C}_K\}$, we say \mathcal{G} is *cluster-scrambling* (w.r.t. \mathcal{C}) if for any pair of vertices $(v_{p_1}, v_{p_2}) \subset \mathcal{C}_p$, there exists a vertex $v_k \in \mathcal{V}$, such that both (v_k, v_{p_1}) and (v_k, v_{p_2}) are in \mathcal{E} .

In [9], we extended ergodicity coefficient [19] and Hajnal diameter [20] to the clustering case and defined the cluster ergodicity coefficient (w.r.t \mathcal{C}) of a stochastic matrix A as

$$\mu_{\mathcal{C}}(A) = \min_{p=1, \dots, K} \min_{i, j \in \mathcal{C}_p} \sum_{k=1}^N \min(A_{ik}, A_{jk})$$

It can be seen that $\mu_{\mathcal{C}}(A) \in [0, 1]$ and A is cluster-scrambling (w.r.t. \mathcal{C}) if and only if $\mu_{\mathcal{C}}(A) > 0$. Furthermore, we say A is δ -cluster-scrambling if $\mu_{\mathcal{C}}(A) > \delta$.

Hajnal diameter proposed in [20] was also generalized to the cluster case:

Definition 4: [9] For a given clustering \mathcal{C} and a matrix A , which has row vectors A_1, A_2, \dots, A_n , define the cluster Hajnal diameter as $\Delta_{\mathcal{C}}(A) = \max_{p=1, \dots, K} \max_{i, j \in \mathcal{C}_p} \|A_i - A_j\|$ for some norm $\|\cdot\|$.

Remark 1: In [9], we have generalized Hajnal inequality to the following cluster Hajnal inequality, i.e.

$$\Delta_{\mathcal{C}}(AB) \leq (1 - \mu_{\mathcal{C}}(A))\Delta_{\mathcal{C}}(B) \quad (2)$$

where A is a stochastic matrix and B is a matrix or a vector.

This inequality indicates that the cluster Hajnal diameter of AB strictly decreases when compared with B , if A is cluster scrambling, i.e., $\mu_{\mathcal{C}}(A) > 0$.

III. CLUSTER CONSENSUS ANALYSIS

Let $x(t) = [x_1(t), \dots, x_n(t)]^\top \in \mathbb{R}^n$ denote the state trajectory of all agents and $I(t) = [I_1(t), \dots, I_n(t)]^\top$. The system (1) can be written in the following impact form:

$$\dot{x}(t) = L(t)x(t) + I(t) \quad (3)$$

Definition 5: System (3) is said to be *intra-cluster synchronized* if any solution $x(t)$ satisfies $\lim_{t \rightarrow \infty} |x_i(t) - x_{i'}(t)| = 0$ for all $i, i' \in \mathcal{C}_p$ and $p = 1, \dots, K$; *inter-cluster separated* if $\limsup_{t \rightarrow \infty} \min_{i \in \mathcal{C}_k, j \in \mathcal{C}_l, k \neq l} |x_i(t) - x_j(t)| > 0$. The system (1) realizes *cluster consensus* if each solution $x(t)$ is bounded, intra-cluster synchronized and inter-cluster separated.

It can be seen that intra-cluster synchronization is equivalent to the stability of the following *cluster consensus subspace* w.r.t. the clustering \mathcal{C} :

$$\mathcal{S}_{\mathcal{C}} = \left\{ x \in \mathbb{R}^n : x_i = x_j, \text{ if } i, j \in \mathcal{C}_p, p = 1, \dots, K \right\}$$

A prerequisite requirement for cluster consensus is that $\mathcal{S}_{\mathcal{C}}$ should be invariant through (1).

Lemma 1: If the following conditions are satisfied: (1). $I_i(t) = I_j(t)$ for all $i, j \in \mathcal{C}_p$ and all $p = 1, \dots, K$; (2). for each pair of p and q , $\sum_{j \in \mathcal{C}_q} L_{ij}(t)$ is identical w.r.t. all $i \in \mathcal{C}_p$ at any time t , then the cluster-consensus subspace is invariant through (1).

The proof is similar to Lemma 3 in [9] and is omitted.

The input is said to be *intra-cluster identical* if the condition (1) in Lemma 1 is satisfied, and the matrix $L(t)$ has *inter-cluster common influence* if condition (2) is satisfied.

Denote $B_{pq}(t) \triangleq \sum_{j \in \mathcal{C}_q} L_{ij}(t)$ w.r.t. all $i \in \mathcal{C}_p$ at any time t and call $B(t) = [B_{pq}(t)]$ the *inter-cluster common influence matrix*.

A. Theoretical results

In the following, we assume

- \mathcal{A}_1 : For any $t \geq t_0$, $L(t)$ is Metzler matrix with all row sums zeros and the elements $L_{ij}(t) \geq 0$ are piecewise continuous;
- \mathcal{A}_2 (*inter-cluster common influence*): For any $t \geq t_0$, there exists a zero row sum Metzler matrix $B(t) = [B_{p,q}(t)]_{p,q=1}^K \in \mathbb{R}^{K,K}$, where

$$\sum_{j \in \mathcal{C}_q} L_{ij}(t) = B_{p,q}(t), \quad i \in \mathcal{C}_p, \quad p, q = 1, \dots, K \quad (4)$$

We highlight that the concept *inter-cluster common influence* coincides with the concept *ordinary lumpability* in Markov chain theory [21].

\mathcal{A}_3 : For any i , $I_i(t)$ is piecewise continuous, both $I_i(t)$ and $\int_{t_0}^t I_i(s)ds$ are bounded, and $I_i(t) = I_j(t) \triangleq \tilde{I}_p(t)$, for all $i, j \in \mathcal{C}_p$, $p = 1, \dots, K$. Let $\tilde{I}(t) = [\tilde{I}_1(t), \dots, \tilde{I}_K(t)]^\top$.

Remark 2: In this paper, we focus on finding the simplest external inputs to guarantee the intra-cluster synchronization and inter-cluster separation. Here the inputs are intra-cluster identical, which counts for intra-cluster synchronization, and inter-cluster different and state-independent, which counts for the inter-cluster separation

Remark 3: If the linearly coupled system can intra-cluster synchronize, the external inputs proposed in this paper can always be used to guarantee the inter-cluster separation, which implies cluster consensus of the linearly coupled systems.

Lemma 2: Suppose $\Phi(t, t_0)$ is the basic solution matrix of the homogeneous system:

$$\dot{v}(t) = L(t)v(t) \quad (5)$$

where $L(t)$ satisfies $\mathcal{A}_1, \mathcal{A}_2$. Then, (1). $\Phi(t, t_0)$ is a stochastic matrix; (2). If $L(t)$ has a δ -cluster-spanning-tree across time interval $[t_0, t_1]$ and $\int_{t_0}^{t_1} L_{ij}(s)ds < M_1$ holds for all $i \neq j$ and some $M_1 > 0$, then $\Phi(t_1, t_0)$ has a δ_1 -cluster-spanning-tree, where $\delta_1 = \min\{1, \delta\}e^{-(n-1)M_1}$.

Proof. 1). Denote $\Phi(t, t_0) = [\Phi_{ij}(t, t_0)] \in \mathbb{R}^{n \times n}$. Since $L(t)$ satisfies assumption \mathcal{A}_2 , if $x(t_0) = \mathbf{1}_n$, then the solution must be $x(t) = \mathbf{1}_n$, which implies each row sum of $\Phi(t, t_0)$ equals 1. Next, we will prove all elements in $\Phi(t, t_0)$ are nonnegative. Note that the i -th column of $\Phi(t, t_0)$ can be regarded as the solution of the following equation:

$$\begin{cases} \dot{x}(t) = L(t)x(t) \\ x(t_0) = e_i^n \end{cases} \quad (6)$$

here e_i^n is an n -dimensional vector whose i -th component is 1 and all the other components are zero. For any $t > t_0$, if $i_0 = i_0(t)$ is the index with $x_{i_0}(t) = \min_{j=1, \dots, n} x_j(t)$, then $\dot{x}_{i_0}(t) = \sum_{j \neq i_0} L_{i_0 j}(x_j(t) - x_{i_0}(t)) \geq 0$. This implies that $\min_j x_j(t)$ is always nondecreasing for all $t > t_0$. Therefore, $x(t) \geq 0$ holds for $t \geq t_0$. Therefore, $\Phi(t, t_0)$ is a stochastic matrix.

2). Consider system (6), since $x_j(t) \geq 0$ holds for all $j = 1, \dots, n$, so $\dot{x}_i(t) \geq L_{ii}(t)x_i(t)$, and $x_i(t) \geq e^{\int_{t_0}^t L_{ii}(s)ds} \geq e^{-(n-1)M_1}$. Meanwhile, we can conclude that $\Phi_{ii}(t_1, t_0)$ is positive. For

each $k \neq i$,

$$\begin{aligned}
x_k(t) &= \sum_{j \neq k} \int_{t_0}^t e^{\int_{\tau}^t L_{kk}(s) ds} L_{kj}(\tau) x_j(\tau) d\tau \\
&\geq \int_{t_0}^t e^{\int_{\tau}^t L_{kk}(s) ds} L_{ki}(\tau) x_i(\tau) d\tau \\
&\geq e^{-(n-1)M_1} \int_{t_0}^t L_{ki}(\tau) d\tau
\end{aligned}$$

So, if $L(t)$ has a δ -edge from vertex j to vertex i across $[t_0, t_1]$, then $\Phi_{ij}(t_1, t_0) \geq e^{-(n-1)M_1} \delta$, which means $\Phi(t_1, t_0)$ has a δ_1 -cluster-spanning-tree

We also present the following assumption for $L(t)$:

\mathcal{A}_4 : There exist an infinite time interval sequence $[t_0, t_1), [t_2, t_3), \dots, [t_{2n}, t_{2n+1}), \dots$, where $t_0 < t_1 \leq t_2 < t_3 \leq \dots$ and a positive sequence $\{\delta_k\}$ which satisfies $\sum_{k=1}^{+\infty} (\delta_k)^{n-1} = +\infty$. And for any $[t_{2k}, t_{2k+1})$, there is a division: $t_{2k} = t_{2k}^0 < t_{2k}^1 < \dots < t_{2k}^{n-1} = t_{2k+1}$, such that $L(t)$ has a δ_k -cluster-spanning-tree across $[t_{2k}^m, t_{2k}^{m+1})$ and $\int_{t_{2k}^m}^{t_{2k}^{m+1}} L_{ij}(s) ds < M_1$, $i \neq j$ with some $M_1 > 0$, $m = 0, \dots, n-2$.

Then, we have the following theorem.

Theorem 1: Assume that $L(t)$ satisfies assumptions $\mathcal{A}_1, \mathcal{A}_2$ and \mathcal{A}_4 . If input $I(t)$ satisfies assumption \mathcal{A}_3 , then system (1) intra-cluster synchronizes.

Proof: Under the assumptions $\mathcal{A}_1, \mathcal{A}_3$, system (1) has a unique solution for any given initial value $x(t_0)$ [22], which has the form $x(t) = \Phi(t, t_0)x(t_0) + \int_{t_0}^t \Phi(t, s)I(s)ds$ with $\Phi(\cdot, \cdot)$ defined in Lemma 2, which implies that $\Phi(t_{2k}^{i+1}, t_{2k}^i)$, $i = 0, \dots, n-2$ are stochastic matrices and have a δ'_k -cluster-spanning-tree with $\delta'_k = \min\{1, \delta_k\}e^{-(n-1)M_1} > 0$. Lemma 1 in [9] indicates that $\Phi(t_{2k+1}, t_{2k})$ is η_k -cluster-scrambling with $\eta_k = (\delta'_k)^{n-1}$. By inequality (2), for any $t \in [t_{2n}, t_{2n+1})$, we have $\Delta_C(\Phi(t, t_0)) \leq \prod_{k=1}^n (1 - \eta_k) \Delta_C(E_n)$.

The assumption $\sum_{k=1}^{+\infty} \delta_k^{n-1} = +\infty$ implies $\sum_{k=1}^{+\infty} \eta_k = +\infty$, which is equivalent to $\lim_{n \rightarrow \infty} \prod_{k=1}^n (1 - \eta_k) = 0$. Hence, $\Delta_C(\Phi(t, t_0))$ converges to zero as time tends to infinity. Since $L(t)$ satisfies the inter-cluster common influence condition, the cluster consensus subspace is an invariant subspace of $\Phi(t, t_0)$. Note that $\Delta_C(I(t)) = 0$. Thus $\Delta_C(\Phi(t, t_0)I(t)) = 0$ for all $t \geq t_0$, which means $\Delta_C(\int_{t_0}^t \Phi(t, s)I(s)ds) = 0$. Therefore, we have $\Delta_C(x(t)) \leq \Delta_C(\Phi(t, t_0)x(t_0))$ converges to zero as $t \rightarrow \infty$. ■

For any vector $z = [z_1, \dots, z_K]^\top$, define

$$\eta(z) = \min_{i \neq j} |z_i - z_j| \quad (7)$$

Theorem 2: Assume that $L(t)$ satisfies assumptions $\mathcal{A}_1, \mathcal{A}_2$ and \mathcal{A}_4 . Let $\Psi(t, t_0)$ be the solution matrix of system $\dot{z}(t) = B(t)z(t)$. If $I(t)$ satisfies assumption \mathcal{A}_3 , $I_i(t)$ does not converge to zero, $i = 1, \dots, n$, and $\limsup_{t \rightarrow \infty} \eta(\int_{t_0}^t \Psi(t, s) \tilde{I}(s) ds) \geq \delta'$ with some $\delta' > 0$, then for almost all initials $x(t_0)$, system (1) reaches cluster consensus.

Proof: We only need to prove that for almost all initials $x(t_0)$, system reaches inter-cluster separation. We introduce the Lyapunov exponent of (5) as follows:

$$\lambda(v) = \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \log \left(\|\Phi(t, t_0)v\| \right).$$

From the Pesin's theory [23], the Lyapunov exponents can only pick finite values and provide a splitting of \mathbb{R}^n . Namely, there is a subspace direct-sum division: $\mathbb{R}^n = \bigoplus_{j=1}^J V_j$, and $\lambda_1 > \dots > \lambda_J$, possibly $J < n$, such that for each $v \in V_j$, $\lambda(v) = \lambda_j$. It's clear that $\lambda_1 = 0$ because $L(t)$ is a Metzler matrix with zero row sum. Let $V = \bigoplus_{j>1} V_j$.

We make the following claim.

Claim: $\mathbb{R}^n = \mathcal{S}_C + V$. This claim is proved in the Appendix. Therefore, for any $x(t_0) \in \mathbb{R}^n$, we can find a vector $y_0 \in \mathcal{S}_C$ such that $x(t_0) - y_0 \in V$. Suppose $y(t)$ is the solution of system: $\dot{y}(t) = L(t)y(t) + I(t)$, $y(t_0) = y_0$. Letting $\delta x(t) = x(t) - y(t)$, then it satisfies $\dot{\delta x}(t) = L(t)\delta x(t)$ with $\delta x(t_0) = y_0 - x(t_0) \in V$, which implies $\lim_{t \rightarrow \infty} \delta x(t) = 0$, i.e. $\lim_{t \rightarrow \infty} [x(t) - y(t)] = 0$.

Thus, instead of $x(t)$, we will discuss whether $y(t) \in \mathcal{S}_C$ inter-cluster separate. Furthermore, we can replace $y(t)$ by a lower-dimensional vector $\tilde{y}(t) \in R^K$ with $\tilde{y}_p(t) = y_i(t)$ for some $i \in \mathcal{C}_p$.

Then, we will discuss the following system:

$$\dot{\tilde{y}}(t) = B(t)\tilde{y}(t) + \tilde{I}(t) \quad (8)$$

where $B(t)$ is defined in assumption \mathcal{A}_2 and $\tilde{I}(t)$ is defined in assumption \mathcal{A}_3 . It is well known that the solution of (8) can be written as

$$\tilde{y}(t) = \Psi(t, t_0)\tilde{y}(t_0) + \int_{t_0}^t \Psi(t, s)\tilde{I}(s)ds$$

Since $\Psi(t, t_0)$ is a stochastic matrix and $\tilde{y}(t_0)$ is bounded, we have $Z_1(t) = \Psi(t, t_0)\tilde{y}(t_0)$ is always bounded. Hence, for any time sequence $\{t_n\}$, $Z_1(t_n)$ has a convergent sub-sequence, still

denoted by $\{t_n\}$. Let $Z_2(t) = \int_{t_0}^t \Psi(t, s) \tilde{I}(s) ds$. From the condition $\limsup_{t \rightarrow \infty} \eta_c(Z_2(t)) \geq \delta'$, one can find a time sequence $\{\hat{t}_n\}_{n=1}^\infty$ such that $\eta_c(Z_2(\hat{t}_n)) \geq \delta'/2$. This implies that each pair of components in $Z_2(\hat{t}_n)$ are not identical. Without loss of generality, suppose $\lim_{n \rightarrow \infty} Z_1(\hat{t}_n) = Z_1^*$, $\lim_{n \rightarrow \infty} Z_2(\hat{t}_n) = Z_2^*$; otherwise, we can choose a sub-sequence of $\{\hat{t}_n\}$ instead. Obviously, $\eta_c(Z_2^*) \geq \frac{\delta'}{2}$. Furthermore, for almost every initial value $x(t_0)$, associated with almost every $\tilde{y}(t_0)$, $Z_1(\hat{t}_n)\tilde{y}(t_0) + Z_2(\hat{t}_n)$ has no pair of components identical when n is sufficiently large. Therefore, for almost every initial value $x(t_0)$, when n is sufficiently large, $\tilde{y}(\hat{t}_n)$ has no identical components, which implies that the state of one cluster in $y(\hat{t}_n)$ are not identical to another. ■

In the following corollaries, we suppose the inputs among different clusters differ by proportionality constants,

$$I_i(t) = \alpha_p u(t), \quad \text{if } i \in \mathcal{C}_p \quad (9)$$

$\alpha_1, \dots, \alpha_K$ are constants and $u(t)$ is a scale function. Let $\tilde{\zeta} = [\alpha_1, \dots, \alpha_K]^\top$. This kind of input is easy to construct, as we only need to give a scale input $u(t)$ and $\tilde{\zeta}$.

Corollary 1: Suppose $L(t)$ satisfies $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4$ and $I(t)$ has form (9) with \mathcal{A}_3 . Let $\Psi(t, t_0)$ be the solution matrix of $\dot{z}(t) = B(t)z(t)$. If $u(t)$ does not converge to zero and $\limsup_{t \rightarrow \infty} \text{rank}(\int_{t_0}^t \Psi(t, s)u(s)ds) = K$, then for almost all initials $x(t_0)$ and $\tilde{\zeta}$, system (1) can cluster consensus.

Proof: Let $Z_3(t) = \int_{t_0}^t \Psi(t, s)u(s)ds$. From the assumption $\limsup_{t \rightarrow \infty} \text{rank}(\int_{t_0}^t \Psi(t, s)u(s)ds) = K$, one can find a time sequence $\{\hat{t}_n\}_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} Z_3(\hat{t}_n) = Z_3^*$ and $\text{rank}(Z_3^*) = K$. Hence, the set $\{\tilde{\zeta} | \text{there exist } i, j, \text{ such that } [Z_3^* \tilde{\zeta}]_i = [Z_3^* \tilde{\zeta}]_j\}$ is of zero measure in \mathbb{R}^K , which means that for almost every $\tilde{\zeta} \in \mathbb{R}^K$, each pair of components in $Z_3^* \tilde{\zeta}$ are not identical, i.e. $\eta(Z_3^* \tilde{\zeta}) \geq 2\delta'$ with some $\delta' > 0$. Therefore, all conditions in Theorem 2 hold. ■

In the following corollary, we discuss the *static inter-cluster common influence* case, that is \mathcal{A}_2^* : There exists a constant $\mathbb{R}^{K, K}$ stochastic matrix $B = [B_{p,q}]_{p,q=1}^K$, such that

$$\sum_{j \in \mathcal{C}_q} L_{ij}(t) = B_{p,q}, \quad i \in \mathcal{C}_p, \quad p, q = 1, \dots, K \quad (10)$$

Corollary 2: Suppose $L(t)$ satisfies the assumptions $\mathcal{A}_1, \mathcal{A}_2^*, \mathcal{A}_4$ and $I(t)$ satisfies assumption \mathcal{A}_3 and (9). If $u(t)$ does not converge to zero, then for almost all initials $x(t_0)$ and $\tilde{\zeta}$, the solution of system (1) can cluster consensus.

Proof: Note that $e^{B(t-t_0)}$ is the solution matrix of $\dot{z}(t) = Bz(t)$. According to Corollary 1, we only need to prove $\limsup_{t \rightarrow \infty} \text{rank}(\int_{t_0}^t e^{B(t-s)} u(s) ds) = K$. Suppose the eigenvalues of B are μ_1, \dots, μ_K (possibly overlap), then the eigenvalues of $W_2(t)$ should be $F_i(t) = \int_{t_0}^t e^{\mu_i(t-s)} u(s) ds$, $i = 1, \dots, K$. From the assumptions, $u(t)$ should be positive and negative intermittently with respect to time. Hence, there exists $\{\hat{t}_n\}_{n=1}^\infty$ such that $\lim_{n \rightarrow \infty} F_i(\hat{t}_n) = F_1^* \neq 0, i = 1, \dots, K$. ■

Remark 4: In Corollary 2, the assumption of existence of a static inter-cluster common influence matrix B can be weakened to be in the form of $a(t)B$, with a scalar function $a(t)$. The sufficient condition can be easily derived from the above analysis.

Remark 5: The realization of the inputs $I_i(t)$ is technical: First, to realize inter-cluster separation, $I_i(t)$ cannot converge to zero asymptotically; otherwise, its influence to the system could disappear; Second, $\int_{t_0}^t I_i(s) ds$ should be bounded to guarantee boundedness of the system, which implies that $I_i(t)$ should be positive and negative intermittently with respect to time, which results in the algebraic difference (without absolute values) between the states in different clusters is positive and negative intermittently as well. In particular, it can be proved that the inter-cluster absolute difference has infinite zeros, which implies that the algebraic values cross zeros infinitely (the proof has not been shown in this paper due to the space limit). For example, $I_i(t) = \alpha_i \sin(t)$ in the following.

IV. SIMULATIONS

In this section, two numerical simulations are provided to illustrate the validity of the proposed theoretic results. The graph models considered here come from [24]. We consider two time-varying graph models: one is so called *p-nearest-neighborhood regular graph*. The graph has N nodes, ordered by $\{1, \dots, N\}$. Each node i has $2r$ neighbors: $\{(i+j) \bmod N : j = \pm 1, \dots, \pm r\}$, where \bmod denotes modular operator. The nodes are divided into K groups: $\mathcal{C}_k = \{i : i \bmod K = k\}, k = 0, \dots, K-1$, where $N \bmod K = 0$. The other one is *bipartite random graph*. N (an even integer) nodes are divided into two groups and each group has $N/2$ nodes. Each node has m neighbors, among which there are $s < m$ neighbors in the same group and the remaining in another group. The neighbors are chosen with equal probability.

In these two examples, nodes are divided into two clusters, colored by red and blue respectively.

The non-identical inputs are defined as :

$$I_p(t) = \alpha_p \sin(t), p = 1, 2.$$

corresponding to each group with α_1, α_2 are randomly selected in $[0, 10]$ with the uniform distribution. Intra-cluster synchronization is measured by difference of states in same clusters:

$$\Delta_c(x(t)) = \max_p \max_{i, i' \in \mathcal{C}_p} |x_i(t) - x_{i'}(t)|$$

Inter-cluster separation is measured by $\eta_c(x(t))$ defined in (7).

Realize these two graph models respectively. We take a switching time sequence $\{t_k\}_{k=0}^{+\infty}$ as a partition of $[0, +\infty)$ with $0 = t_0 < t_1 < \dots$. Denote $\Delta t_i = t_i - t_{i-1}$, and the switching time interval Δt_i is uniformly distributed on $(0, 1)$.

At every switching time, the graph topology stochastically choose from these two topologies given in the top panels of Figs. 1 (a) and (b) respectively. For $t \in [t_{k-1}, t_k)$, take $L_{ij}(t) = \sin(\frac{\pi(t-t_{k-1})}{\Delta t_k})$ if j is a neighbor of i ; otherwise, $L_{ij}(t) = 0$ and $L_{ii}(t) = -\sum_{j \neq i} L_{ij}(t)$. Pick $\delta = 1$. $L(t)$ has δ -cluster-spanning-trees across $[t_i, t_{i+3})$. Furthermore, the input $u(t) = \sin(t)$ and its integral are both bounded. Meanwhile, we notice that the inter-cluster common influence matrix satisfies: $B(t) = \sin(\frac{\pi(t-t_{k-1})}{\Delta t_k})B$ when $t_{k-1} \leq t < t_k$. Denote $B(t) = b(t)B$. $\Psi(t, t_0) = e^{\int_{t_0}^t b(s)dsB}$ is the solution matrix of system $\dot{z}(t) = B(t)z(t)$.

Therefore, all conditions in Theorems 1 and 2 are satisfied. Choose the initial values randomly. In Fig.1(a) and (b), the dynamical behaviors of the states are plotted, while nodes in the same clusters are plotted in same color. In the bottom panels of Fig.1 (a) and (b), the blue, red and green curves respectively show the dynamical behaviors of $\eta_c(x(t))$, $\Delta_c(x(t))$ and $\eta_c(x(t)) + \eta_c(v(t))$ with respect to the time-varying topologies, where $v(t) \triangleq \dot{x}(t)$. All of them show that the cluster consensus is reached. Please note that according to the arguments before, $I_p(t) = \alpha_p \sin(t)$ takes negative and positive values intermittently so that $\int_{t_0}^t I_i(s)ds$ is bounded with respect to t , but never converges to zero. This implies that there are infinite zeros of η_c since its algebraic values cross zeros infinite times, as shown in the third panels of Fig 1 (a,b) respectively.

V. CONCLUSIONS

In this paper, we have investigated cluster consensus problem in continuous-time networks of multi-agents with non-identical inter-cluster inputs. Sufficient conditions for cluster consensus

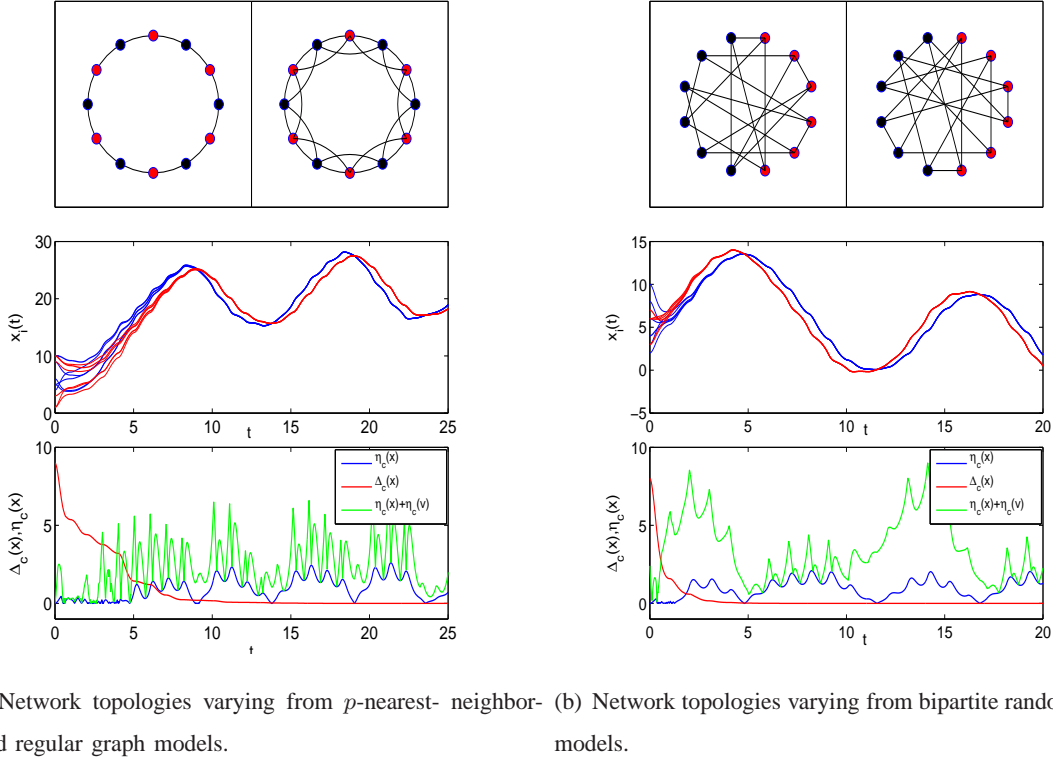


Fig. 1. The dynamics of states $\{x_i(t)\}$ and measures $\Delta_c(x(t)), \eta_c(x(t))$. Red and blue nodes show the two clusters of nodes respectively.

for systems with time-varying graph topologies were derived. By defining cluster consensus subspace, cluster consensus problem was transformed to the stability of the cluster consensus subspace under inter-cluster common influence condition. The separation among states in different clusters were guaranteed by external inputs. From algebraic graph theory, it was indicated that the receiving same amount of information for agents in the same cluster is a doorsill for the complete synchronization of agents in the same cluster. The effectiveness of the proposed theoretical results were demonstrated by numerical simulations.

VI. APPENDIX

Proof of Claim 1: Define a $\mathbb{R}^{n,n}$ nonsingular matrix $P = [P_1, \dots, P_n]$ with the first K column vectors composed of a basis of \mathcal{S}_C . Thus, let

$$\hat{L}(t) \triangleq P^{-1}L(t)P = \begin{bmatrix} B(t) & \hat{L}_{1,2}(t) \\ O & \hat{L}_{2,2}(t) \end{bmatrix},$$

$$\hat{\Phi}(t, t_0) \triangleq P^{-1}\Phi(t, t_0)P = \begin{bmatrix} \Psi(t, t_0) & \hat{\Phi}_{1,2}(t, t_0) \\ O & \hat{\Phi}_{2,2}(t, t_0) \end{bmatrix},$$

where $\Psi(t, t_0)$ is the solution matrix of system $\dot{x}(t) = B(t)x(t)$. We define the *projection radius* (w.r.t. C) of $\Phi(t, t_0)$ as follows:

$$\rho_C(\Phi(\cdot, t_0)) = \overline{\lim}_{t \rightarrow \infty} \left\{ \|\hat{\Phi}_{2,2}(t, t_0)\| \right\}^{1/t}$$

and the *cluster Hajnal diameter* (w.r.t. C) of $\Phi(t, t_0)$ as follows:

$$\Delta_C(\Phi(\cdot, t_0)) = \overline{\lim}_{t \rightarrow \infty} \left\{ \Delta_C(\Phi(t, t_0)) \right\}^{1/t}$$

for some norm $\|\cdot\|$ that is induced by vector norm. Select one single row in $\Phi(t, t_0)$ from each cluster and compose these rows into a matrix, denoted by H . Let $G = [P_1, \dots, P_K]$. It can be seen that the rows of GH corresponding to the same cluster are identical. Then, we have

$$\begin{aligned} \|\Phi(t, t_0) - GH\| &= \|P^{-1}\Phi(t, t_0)P - \begin{bmatrix} E \\ O \end{bmatrix} HP\| \\ &= \left\| \begin{bmatrix} Y & Z \\ O & \hat{\Phi}_{22}(t, t_0) \end{bmatrix} \right\|, \end{aligned}$$

which implies $\rho_C(\Phi(\cdot, t_0)) \leq \Delta_C(\Phi(\cdot, t_0))$. In Theorem 1, $\Delta_C(\Phi(\cdot, t_0)) < 1$ has been proved. Thus, $\rho_C(\Phi(\cdot, t_0)) < 1$, which means $\hat{\Phi}_{2,2}(t, t_0)$ converges to zero matrix exponentially.

It can be seen that $\hat{\Phi}(t, t_0)$ is the solution matrix of system $\dot{w}(t) = P^{-1}L(t)Pw(t)$. Consider the block form of vector $w(t) = \hat{\Phi}(t, t_0)w(t_0)$:

$$\begin{cases} w_1(t) = \Psi(t, t_0)w_1(t_0) + \hat{\Phi}_{1,2}(t, t_0)w_2(t_0) \\ w_2(t) = \hat{\Phi}_{2,2}(t, t_0)w_2(t_0). \end{cases} \quad (11)$$

$\rho_C(\Phi(\cdot, t_0)) < 1$ implies that $w_2(t)$ converges to $\mathbf{0}$ exponentially. Then define the operators $R_1 = \lim_{t \rightarrow \infty} \Psi^{-1}(t, t_0) \hat{\Phi}_{1,2}(t, t_0)$. It can be verified that R_1 is well defined. Consider a subspace of \mathbb{R}^n : $\tilde{V} = \left\{ [z^\top, v^\top]^\top \in \mathbb{R}^n : z = -R_1 v \right\}$.

For any n -dimensional vector $w_0 = [z_0, v_0]^\top$, we rewrite w_0 as a sum of $w_0^1 + w_0^2$ with $w_0^1 = [z_0^1, \mathbf{0}]^\top$, $w_0^2 = [z_0^2, v_0]^\top$. If we take $w(t_0) = w_0^2$ and pick z_0^2 such that $w_0^2 \in \tilde{V}$, then $w(t)$ converges to $\mathbf{0}$ exponentially. That is, $PQw_0^2 \in V$. On the other hand, PQw_0^1 corresponds a vector in \mathcal{S}_C . Therefore, for any n -dimensional vector x_0 , we can find w_0 , such that $x_0 = PQw_0 = PQw_0^1 + PQw_0^2 \in \mathcal{S}_C + V$.

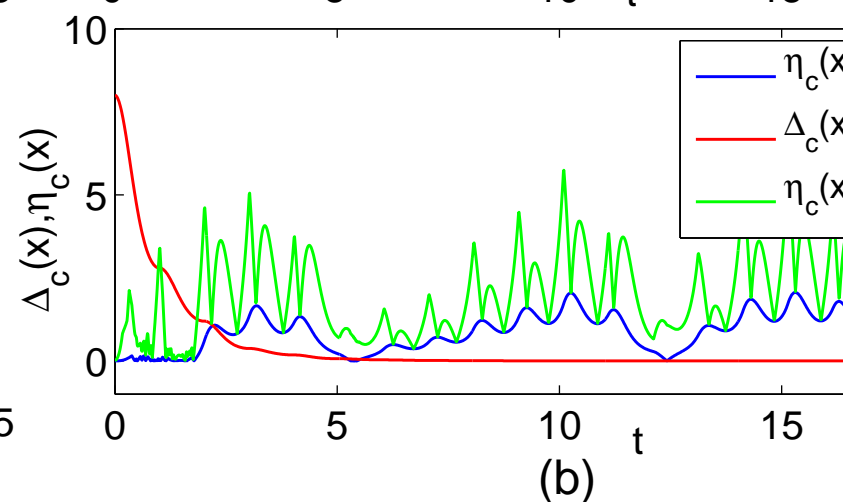
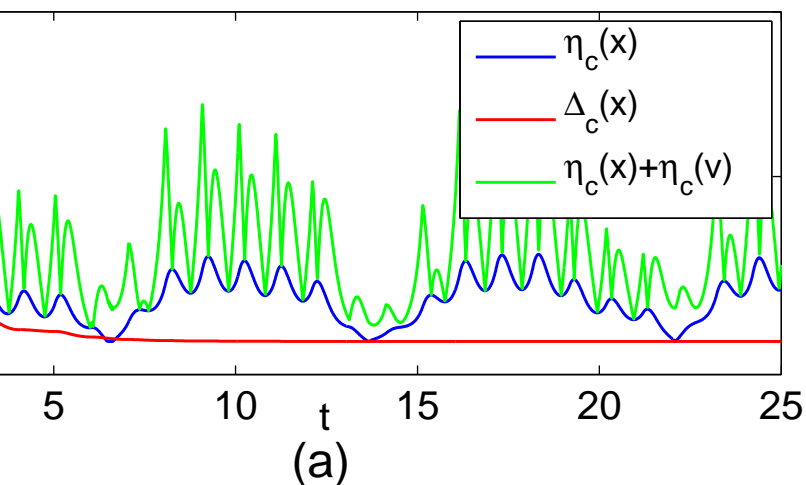
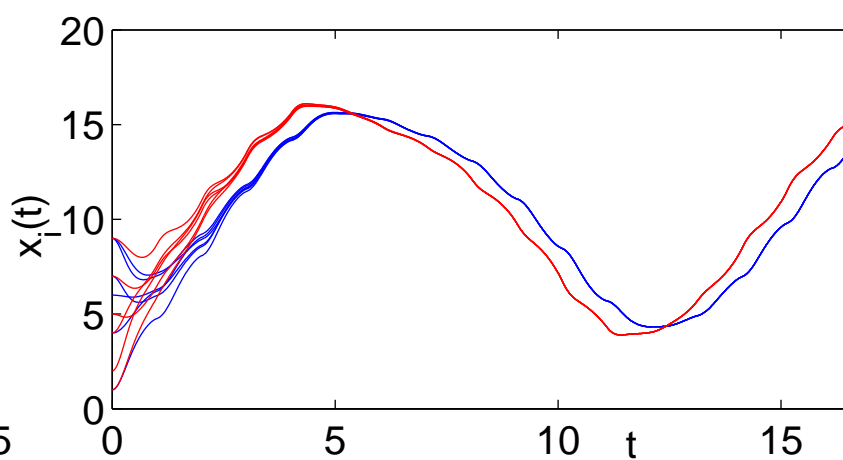
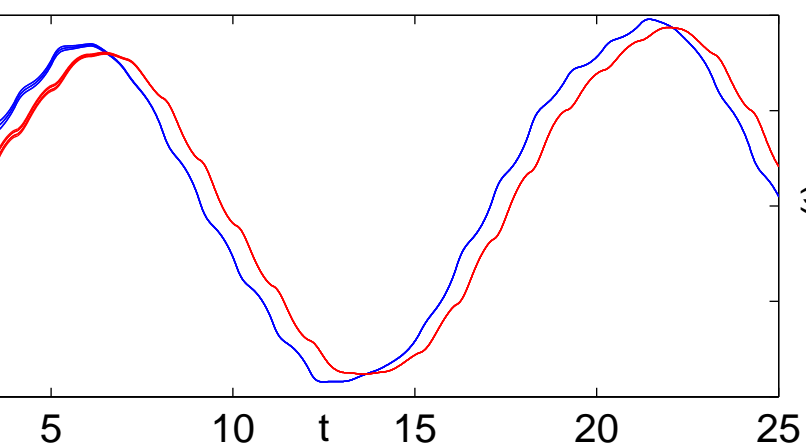
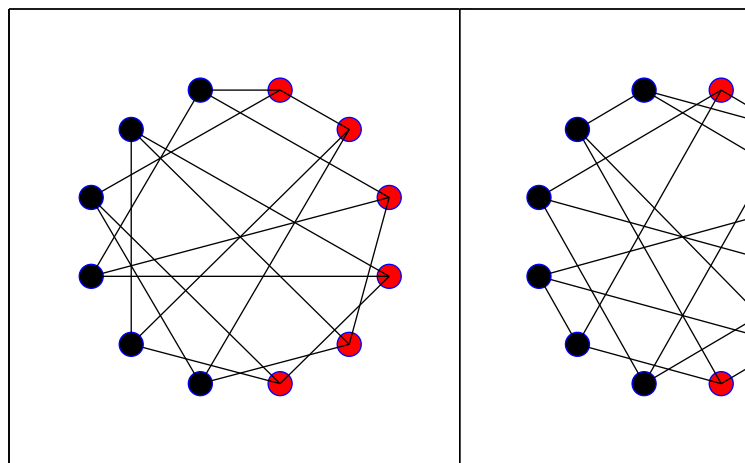
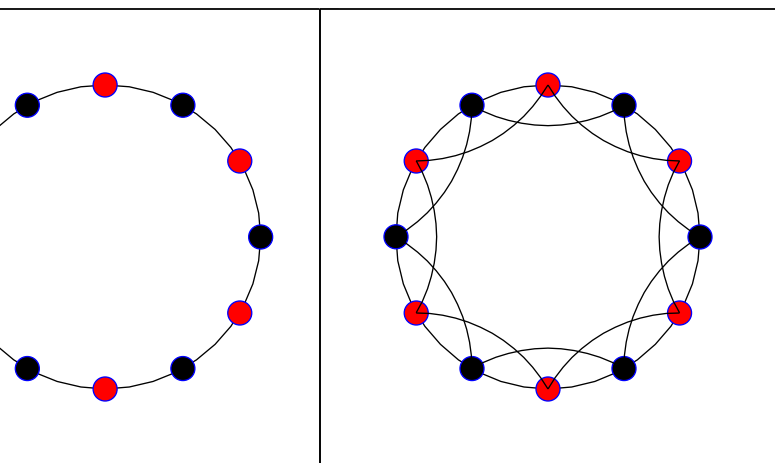
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(a)

(b)